Conditional fractional strong matching preclusion of pancake graphs via a recursive isoperimetric inequality

Kyle Vedder

Panal Capital, San Francisco, United States of America

Abstract

We determine the conditional fractional strong matching preclusion number of the pancake graphs and classify all optimal mixed vertex/edge fault sets. This invariant of a graph G is the minimum size of a set F of vertices and edges such that $G - F$ has neither a fractional perfect matching nor an isolated vertex. For the pancake graph P_n, the Cayley graph of the symmetric group generated by all prefix reversals, we prove $\mathrm{cfsmp}(P_4) = 3$ and $\mathrm{cfsmp}(P_n) = 2n - 4$ for $n \geq 5$. The proof introduces a recursive isoperimetric inequality for the pancake Cayley graphs: for $n \geq 5$, every vertex subset X with $|X| \geq 2$ has isoperimetric cost $e(X) + |\partial X| - |X| + 1 \geq 2n - 4$, with equality precisely for pairs of vertices at distance at most two. This inequality converts every fractional Berge–Tutte obstruction in a faulty pancake graph into a sharp boundary count. As a consequence, for $n \geq 5$ the optimal fault sets are exactly the sets obtained from an induced path u–w–v by deleting, for each neighbor y of u or v other than w, either the vertex y itself or its unique edge toward $\{u, v\}$; in particular, P_n is conditionally fractional strongly super matched. This settles the case of the ordinary pancake graphs left open by Gupta, Cheng, and Lipták, and their composition theorem for pancake-like graphs then applies with copies of P_n as building blocks.

Keywords: pancake graph, matching preclusion, fractional perfect matching, Cayley graph, isoperimetric inequality, interconnection network
2020 MSC: 05C70, 05C25, 05C35, 68R10

Email address: `kyle@panal.capital` (Kyle Vedder)

1. Introduction

The perfect matching structure of a graph is among the first casualties of vertex and edge faults, and the matching preclusion numbers quantify exactly how robust that structure is. These are graph invariants in the tradition of exact extremal graph theory: one asks for the minimum number of deletions that destroys a matching property, and, in the strongest form, for the complete list of minimum deletion sets. The same invariants are studied in the interconnection network literature as fault-tolerance measures, since a network whose surviving nodes can no longer be paired has lost a basic communication primitive. Both viewpoints inform this paper: the objects are Cayley graphs of symmetric groups, the question is an exact extremal one, and the answer takes the form of a value together with a structural classification of the extremal configurations.

The matching preclusion number, introduced by Brigham, Harary, Violin, and Yellen [2], is the minimum number of edge deletions leaving a graph with no perfect matching. Since many highly symmetric graphs are most cheaply defeated by isolating a single vertex, Cheng, Lesniak, Lipman, and Lipták [5] introduced the conditional version, which forbids isolated vertices in the faulty graph and thereby exposes the next layer of obstructions. Park and Ihm [17] introduced the strong version, in which vertices and edges may both fail; Mao and Cheng [16] survey these variants.

Fractional relaxations of these invariants were introduced by Liu and Liu [13]: perfect matchings are replaced by fractional perfect matchings, that is, edge weight functions $w : E(G) \to [0, 1]$ with unit total weight at every vertex. The general theory of the fractional invariants was developed by Lin and Zhang [12] through the perfect matching polytope, and by Zou, Mao, Wang, and Cheng [20], who proved sharp general bounds with extremal characterizations. For the pancake family, Cheng, Hu, Jia, Lipták, Scholten, and Voss [3] computed the ordinary and conditional matching preclusion numbers of pancake and burnt pancake graphs, and Ma, Mao, Cheng, and Melekian [15] computed the fractional and fractional strong matching preclusion numbers of both families; in particular, $\mathrm{fsmp}(P_n) = n-1$, and P_n is fractional strongly super matched for $n \geq 5$.

The conditional and fractional strands were combined by Hu, Tian, Meng, and Yang [10] for edge faults in torus networks, and then in full generality by Gupta, Cheng, and Lipták [7, 8], who defined the conditional fractional strong matching preclusion number $\mathrm{cfsmp}(G)$: the minimum size of a mixed vertex/edge fault set F such that $G - F$ has no fractional perfect matching and no isolated vertex. They proved $\mathrm{cfsmp}(B_n) = 2n - 2$ for the burnt pancake graphs B_n and showed that every optimal fault set is trivial (a property they call *conditionally fractional strongly super matched*), and they proved a composition theorem for pancake-like graphs: supermatchedness propagates to a recursively assembled graph provided every building block is both fractional strongly super matched and conditionally fractional strongly super matched.

These results do not settle the ordinary pancake graphs. The composition theorem requires the conditional supermatchedness of the building blocks, and for the ordinary pancake family that hypothesis was precisely what was missing: their program reduces P_n to a base-case verification for P_5, analogous to the computer check they performed for B_3. That check already took about one day over 1,302,609 candidate fault sets, each requiring a linear-programming feasibility test, and they single out the P_5 computation as beyond a conventional computer: "This may be feasible by using a computer cluster" [7, p. 433].

This paper settles the ordinary pancake family, for all dimensions simultaneously, with a proof that involves no fault-set enumeration at all. The only computational ingredient is a table of subset minima in the 24-vertex graph P_4, obtained by counting edges and boundary vertices over its 2^{24} vertex subsets—a computation of a few seconds, with no linear programming. The first main result gives the exact values for every $n \geq 4$; the degenerate cases $n \leq 3$ are dispatched in Section 8.

Theorem 1.1. $\operatorname{cfsmp}(P_4) = 3$, *and*

$$\operatorname{cfsmp}(P_n) = 2n - 4 \quad \textit{for every } n \geq 5.$$

The second main result is a complete classification of the optimal fault sets for $n \geq 5$. Fix an induced path u–w–v in P_n and put $B = (N(u) \cup N(v)) \setminus \{w\}$. For a subset $Z \subseteq B$, let

$$F(u, w, v; Z) = Z \ \cup \ \{xy : \ y \in B \setminus Z, \ x \in \{u, v\}, \ xy \in E(P_n)\}.$$

Here the elements of Z are vertex faults and the displayed edges are edge faults; no displayed edge is incident with a vertex of Z.

Theorem 1.2. *Let $n \geq 5$. The optimal conditional fractional strong matching preclusion sets of P_n are exactly the sets $F(u, w, v; Z)$, where u–w–v ranges over the induced paths of length two in P_n and Z ranges over the subsets of B. In particular, every optimal set leaves two vertices of degree one adjacent to a common surviving vertex, so P_n is conditionally fractional strongly super matched.*

Two features of Theorem 1.2 deserve emphasis. First, the pure edge-deletion construction (the case $Z = \emptyset$) is not the only optimal construction: any boundary vertex fault may replace the corresponding boundary edge fault, and these mixed sets exhaust all optimal sets. Second, the classification is strictly finer than supermatchedness: it identifies the extremal configurations exactly, not merely their local shape near the two leaves.

The engine behind both theorems is an isoperimetric inequality tailored to the fractional perfect matching criterion. For $X \subseteq V(P_n)$, write $e(X)$ for the number of edges inside X, write ∂X for the set of vertices outside X with a neighbor in X, and define the *isoperimetric cost*

$$\lambda_n(X) = e_{P_n}(X) + |\partial_{P_n} X| - |X| + 1.$$

A short argument (Lemma 3.1) shows that every conditional fractional strong matching preclusion set F admits a witness set X with $|X| \geq 2$ and $|F| \geq \lambda_n(X)$: every fractional Berge–Tutte obstruction pays at least the isoperimetric cost of its witness set. The lower bound in Theorem 1.1 and the classification in Theorem 1.2 then follow from the third main result, whose equality part we regard as a contribution of independent interest.

Theorem 1.3. *Let $n \geq 5$ and let $X \subseteq V(P_n)$ with $|X| \geq 2$.*

- (a) $\lambda_n(X) \geq 2n - 4$.
- (b) $\lambda_n(X) = 2n - 4$ *if and only if* $X = \{u, v\}$, *where u and v are adjacent or have a common neighbor in P_n.*

The proof of Theorem 1.3 is a recursion over the standard decomposition of P_n into n vertex-disjoint copies of P_{n-1} joined by a perfect matching of prefix-reversal edges. An exact identity (Lemma 4.1) expresses $\lambda_n(X)$ in terms of the values of λ_{n-1} on the pieces of X plus nonnegative cross terms, and the girth structure of the pancake graphs—they contain no triangles and no 4-cycles—eliminates the tight configurations. The recursion bottoms out in the exact P_4 table.

Theorem 1.2 also supplies the hypothesis that was missing from the composition theorem of Gupta, Cheng, and Lipták. As a consequence, every pancake-like graph assembled from copies of P_m with $m \geq 5$ (or from a mixture of such copies and burnt pancake graphs B_{m-1}) is fractional strongly super matched and conditionally fractional strongly super matched; see Corollary 7.2.

The broader preclusion landscape has continued to develop around these questions. Abdallah and Cheng [1] determined the conditional strong matching preclusion number of the pancake graph in the integral model. Luan, Lu, and Zhang [14] studied fractional matching preclusion of Cartesian products. Wu, Ba, and Zhang [19] and Hu, Ren, and Yang [9] developed integer k-matching preclusion, the latter solving the conditional version for torus networks. Li, Liu, Li, Cheng, and Mao [11] studied conditional fractional (strong) matching preclusion for several general graph classes, and Cheng, Lipták, and Mazza [4] introduced a level-2 matching preclusion framework for regular interconnection networks with applications to pancake graphs. None of these works gives the conditional fractional strong value or the optimal mixed-fault classification for the ordinary pancake graphs.

The paper is organized as follows. Section 2 collects definitions and the two short-cycle lemmas for pancake graphs. Section 3 converts fractional matching failure into isoperimetric costs. Section 4 proves Theorem 1.3. Section 5 proves Theorem 1.1, and Section 6 proves Theorem 1.2. Section 7 records the composition corollary, and Section 8 closes with contrasts and remarks.

2. Preliminaries

All graphs are finite, simple, and undirected. If $F \subseteq V(G) \cup E(G)$, then $G - F$ is obtained by deleting every vertex in $F \cap V(G)$ and every edge in $F \cap E(G)$ whose endpoints remain.

A *fractional perfect matching* of G is a function $w : E(G) \to [0, 1]$ such that

$$\sum_{e \ni v} w(e) = 1$$

for every $v \in V(G)$.

Lemma 2.1 (Fractional perfect matching criterion)**.** *The following standard fractional Berge–Tutte criterion holds [18]: a graph G has a fractional perfect matching if and only if, for every set $S \subseteq V(G)$, the graph $G - S$ has at most $|S|$ isolated vertices.*

Definition 2.2. *A set $F \subseteq V(G) \cup E(G)$ is a* conditional fractional strong matching preclusion set *(CFSMP set) if $G - F$ has no fractional perfect matching and has no isolated vertex. The minimum size of such an F is denoted* cfsmp(G).

Following Gupta, Cheng, and Lipták [8], an optimal CFSMP set F is *trivial* if $G - F$ contains vertices u, v, w such that

$$\deg_{G-F}(u) = \deg_{G-F}(v) = 1 \quad \text{and} \quad uw, vw \in E(G - F).$$

A graph is *conditionally fractional strongly super matched* if every optimal CFSMP set is trivial.

We also need the unconditional strong invariant [13]. A set $F \subseteq V(G) \cup E(G)$ is a *fractional strong matching preclusion set* (FSMP set) if $G - F$ has no fractional perfect matching, and $\operatorname{fsmp}(G)$ is the minimum size of an FSMP set. An optimal FSMP set F is *trivial* if $G - F$ has an isolated vertex, and G is *fractional strongly super matched* if every optimal FSMP set is trivial.

For a graph G and a set $X \subseteq V(G)$, write $e_G(X)$ for the number of edges with both endpoints in X, and write

$$\partial_G X = N_G(X) \setminus X$$

for the open vertex boundary of X. Define

$$\lambda_G(X) = e_G(X) + |\partial_G X| - |X| + 1,$$

the *isoperimetric cost* of X in G; Lemma 3.1 explains the name.

2.1. Pancake graphs

For $2 \leq k \leq n$, let r_k be the permutation of positions that reverses the first k positions and fixes all later positions. The pancake graph is

$$P_n = \operatorname{Cay}(\mathfrak{S}_n, \{r_2, r_3, \ldots, r_n\}).$$

Thus two permutations are adjacent if one is obtained from the other by a prefix reversal. The graph P_n is $(n-1)$-regular.

The proofs below use only two structural facts about short cycles; both are special cases of the girth computation of Compeau [6], and we include the short proofs to keep the paper self-contained.

Lemma 2.3. *The pancake graph P_n is triangle-free.*

Proof. A triangle in the Cayley graph would give a non-backtracking equality $r_a r_b = r_c$ with $a \neq b$. Let $t = \max(a, b)$. The product $r_a r_b$ fixes every position greater than t and moves position t, so if it is a prefix reversal then it must be r_t.

If $a = t > b$, then $(r_a r_b)(1) = t + 1 - b$, whereas $r_t(1) = t$; this would force $b = 1$, impossible. If $b = t > a$, then $(r_a r_b)(t) = a$, whereas $r_t(t) = 1$; this would force $a = 1$, again impossible. Thus no such equality exists. □

Lemma 2.4. *The pancake graph P_n is C_4-free.*

Proof. It is enough to show that, whenever $a \neq b$, the product $r_a r_b$ determines the ordered pair (a, b).

Let $t = \max(a, b)$. The product $r_a r_b$ fixes every position greater than t and moves position t, so t is determined by the product.

If $a = t > b$, then

$$(r_a r_b)(t) = 1 \quad \text{and} \quad (r_a r_b)(1) = t + 1 - b.$$

Thus $a = t$ and $b = t + 1 - (r_a r_b)(1)$ are recovered.

If $b = t > a$, then

$$(r_a r_b)(t) = a \neq 1,$$

so $b = t$ and $a = (r_a r_b)(t)$ are recovered.

A 4-cycle in the Cayley graph would give a non-backtracking equality $r_a r_b = r_d r_c$ with $a \neq b$ and $c \neq d$. The uniqueness just proved forces $(a, b) = (d, c)$, so the supposed 4-cycle backtracks and repeats a vertex. This is impossible. Hence P_n is C_4-free. □

In particular, in P_n two adjacent vertices have no common neighbor, and two nonadjacent vertices have at most one common neighbor.

3. From Berge–Tutte witnesses to isoperimetric costs

The next two lemmas hold in an arbitrary graph. The first shows that every fractional Berge–Tutte obstruction in a faulty graph forces the fault set to pay the isoperimetric cost λ_G of a witness set; the second records exactly what happens when that cost is paid with no surplus, which is the leverage behind the classification in Theorem 1.2.

Lemma 3.1 (Witness cost)**.** *Let G be a graph and let $F \subseteq V(G) \cup E(G)$ be a conditional fractional strong matching preclusion set. Then there is a set $X \subseteq V(G)$ with $|X| \geq 2$ such that*

$$|F| \geq \lambda_G(X).$$

Proof. Put $H = G - F$. By Lemma 2.1, there is a set $S \subseteq V(H)$ such that $H - S$ has a set X of isolated vertices with $|X| > |S|$. Since H has no isolated vertex, $|X| \neq 1$: if $|X| = 1$, then $S = \emptyset$, and the single vertex of X would already be isolated in H. Hence $|X| \geq 2$.

Every edge of $G[X]$ must be an edge fault, since all vertices of X survive in $H - S$ and are isolated there. Thus F contains at least $e_G(X)$ internal edge faults.

Now let $y \in \partial_G X \setminus S$. If y is a vertex fault, count that fault. If y is not a vertex fault, then at least one edge between y and X must be an edge fault, otherwise y would be adjacent in $H - S$ to a vertex of X. These counted faults are distinct for distinct boundary vertices y, and none is an internal edge of $G[X]$. Therefore

$$|F| \geq e_G(X) + |\partial_G X \setminus S| \geq e_G(X) + |\partial_G X| - |S|.$$

Since $|S| \leq |X| - 1$, this gives

$$|F| \geq e_G(X) + |\partial_G X| - |X| + 1 = \lambda_G(X).$$

□

Lemma 3.2 (Witness cost equality)**.** *Use the notation in the proof of Lemma 3.1: put $H = G - F$, let $S \subseteq V(H)$, and let X be a set of isolated vertices of $H - S$ with $|X| > |S|$. If $|F| = \lambda_G(X)$, then:*

1. *$|S| = |X| - 1$ and $S \subseteq \partial_G X$;*
2. *every edge of $G[X]$ is an edge fault;*
3. *for each $y \in \partial_G X \setminus S$, exactly one fault is associated with y: either y is a vertex fault, or y is not a vertex fault and there is exactly one edge from y to X, which is an edge fault;*
4. *F contains no other faults.*

Proof. The proof of Lemma 3.1 gives

$$|F| \geq e_G(X) + |\partial_G X \setminus S| \geq e_G(X) + |\partial_G X| - |S| \geq e_G(X) + |\partial_G X| - |X| + 1.$$

If $|F| = \lambda_G(X)$, equality holds throughout this chain. The last equality gives $|S| = |X| - 1$. The middle equality gives $|\partial_G X \setminus S| = |\partial_G X| - |S|$, and hence $S \subseteq \partial_G X$.

All edges of $G[X]$ must be edge faults because the vertices of X survive and are isolated in $H - S$. For each $y \in \partial_G X \setminus S$, isolation of X in $H - S$ forces either the vertex fault y, or all edges from y to X to be edge faults. Since equality leaves exactly one counted fault for each such y, this gives the alternatives in (3): in particular, if y is not a vertex fault and had two or more edges to X, then all of them would be edge faults, contributing at least two faults associated with y and forcing $|F| > \lambda_G(X)$. The counted faults have total size $\lambda_G(X) = |F|$, so no additional faults can occur. Note that every element of F contributes to $|F|$ whether or not it is visible in $G - F$; in particular, under equality F contains no redundant edge fault incident with one of its own vertex faults, since such an edge is not among the counted faults. □

4. A recursive isoperimetric inequality

For $X \subseteq V(P_n)$, write

$$\lambda_n(X) = e_{P_n}(X) + |\partial_{P_n} X| - |X| + 1.$$

This section proves Theorem 1.3: for every $n \geq 5$, the isoperimetric cost of every $X \subseteq V(P_n)$ with $|X| \geq 2$ is at least $2n - 4$, and the cost is exactly $2n - 4$ precisely for the pairs of vertices at distance at most two.

4.1. The finite base

The proof uses the following exact table for P_4. It was verified by exhaustive enumeration of the 2^{24} vertex subsets of P_4; the verification program accompanying this paper constructs the Cayley graph and computes $\lambda_4(Y)$ directly for every subset Y. The enumeration only counts internal edges and boundary vertices; no linear programming is involved. The same program verifies the exceptional P_4 fault set used in the proof of Theorem 1.1 (for which Appendix A also gives a self-contained, human-checkable certificate) and, exhaustively over all induced paths and all subsets Z, the non-isolation property of the optimal sets $F(u, w, v; Z)$ in P_5. The program and its output log are provided with the submission as ancillary material and are also available as a public GitHub Gist: `https://gist.github.com/kylevedder/6b577c0be2760d87f6c146da76d68513`.

$\lvert Y\rvert$	1	2	3	4	5	6	7	8	9	10	11	12
$\min \lambda_4(Y)$	3	4	4	5	5	5	5	5	5	4	4	3

$\lvert Y\rvert$	13	14	15	16	17	18	19	20	21	22	23	24
$\min \lambda_4(Y)$	3	3	4	5	6	7	8	9	10	11	12	13

We shall use three consequences:

$$Y \neq \emptyset \quad \Longrightarrow \quad \lambda_4(Y) - 1 \geq 2, \tag{1}$$

$$2 \leq |Y| \leq 6 \quad \Longrightarrow \quad \lambda_4(Y) - 1 \geq 3, \tag{2}$$

and

$$|Y| \geq 2 \quad \Longrightarrow \quad \lambda_4(Y) + |Y| \geq 6. \tag{3}$$

The equality classification also needs the P_5 equality base. We prove it directly after the recursive identity, using only this table and the absence of triangles and 4-cycles.

4.2. The recursive identity

Partition $V(P_n)$ into the n sets

$$V_i = \{\pi \in \mathfrak{S}_n : \text{the last symbol of } \pi \text{ is } i\}.$$

Each $P_n[V_i]$ is naturally a copy of P_{n-1}, generated by $r_2, \dots, r_{n-1}$. For a vertex π, write $\mu(\pi)$ for the *mate* of π, the vertex obtained from π by the full reversal r_n, and for $Y \subseteq V(P_n)$ write $\mu(Y) = \{\mu(\pi) : \pi \in Y\}$. The map μ is an involution without fixed points, so the mate edges $\{\pi, \mu(\pi)\}$ form a perfect matching M of P_n. The mate of a vertex $\pi \in V_i$ lies in the part indexed by the first symbol of π; in particular $\mu(\pi) \notin V_i$, so every edge of M joins different parts.

Let $X_i = X \cap V_i$. Inside the copy $P_n[V_i]$, put

$$A_i = N_{P_n[V_i]}(X_i) \setminus X_i, \qquad A = \bigcup_i A_i.$$

Let $m(X)$ be the number of M-edges with both endpoints in X, and let

$$b(X) = |\mu(X) \setminus (X \cup A)|.$$

For the empty set, the defining formula evaluates to $\lambda_{n-1}(\emptyset) = 0+0-0+1 = 1$, so $\lambda_{n-1}(X_i) - 1 = 0$ when $X_i = \emptyset$.

Lemma 4.1. *For every $X \subseteq V(P_n)$,*

$$\lambda_n(X) = 1 + m(X) + b(X) + \sum_{i=1}^{n} \bigl(\lambda_{n-1}(X_i) - 1\bigr).$$

Proof. The internal edges of X are the internal edges inside the parts, together with the M-edges contained in X. Hence

$$e_{P_n}(X) = \sum_i e_{P_n[V_i]}(X_i) + m(X).$$

Also,

$$\partial_{P_n} X = A \cup (\mu(X) \setminus X).$$

The sets A_i are disjoint, and the only possible overlap between A and $\mu(X) \setminus X$ is removed by the definition of $b(X)$. Therefore

$$|\partial_{P_n} X| = \sum_i |A_i| + b(X).$$

Substituting these two identities into $\lambda_n(X) = e_{P_n}(X) + |\partial_{P_n} X| - |X| + 1$ gives the formula. □

Lemma 4.2 (P_5 equality base)**.** *Let* $X \subseteq V(P_5)$ *with* $|X| \geq 2$*. If* $\lambda_5(X) = 6$*, then* $X = \{u, v\}$ *and the vertices* u, v *are adjacent or have a common neighbor in* P_5*.*

Proof. Decompose P_5 into its five copies V_i of P_4, and let p be the number of nonempty sets $X_i = X \cap V_i$.

If $p = 1$, say $X \subseteq V_i$, then the mate of each vertex of X lies in a part other than V_i, and all such parts are empty. Hence $m(X) = 0$; moreover, the mates are distinct (as μ is injective) and avoid $X \cup A$ (as $A = A_i \subseteq V_i$), so $b(X) = |X|$. Lemma 4.1 gives

$$6 = \lambda_5(X) = \lambda_4(X_i) + |X_i|.$$

By the P_4 table and $|X_i| \geq 2$, this forces $|X_i| = 2$ and $\lambda_4(X_i) = 4$. Write $X_i = \{u, v\}$. Since P_4 is cubic, triangle-free, and C_4-free, the two vertices have no common neighbor when they are adjacent and at most one common neighbor when they are not adjacent. Thus the formula for $\lambda_4(\{u, v\})$ shows that the value 4 occurs exactly when u, v are adjacent or have one common neighbor.

If $p \geq 3$, then (1) gives

$$\lambda_5(X) \geq 1 + 2p \geq 7,$$

contrary to $\lambda_5(X) = 6$. Thus $p = 2$ remains. Let the occupied parts be V_i and V_j.

First suppose some occupied piece, say X_i, has size at least 10. The mate of a vertex of V_i lies in the part indexed by its first symbol. Among the vertices of V_i, at most $3! = 6$ have first symbol j. Therefore at least $|X_i| - 6 \geq 4$ vertices of X_i have mates in empty parts; those mates are distinct and lie outside $X \cup A$ (as $A \subseteq V_i \cup V_j$), so $b(X) \geq 4$. Together with (1) for the two occupied parts, this gives

$$\lambda_5(X) \geq 1 + 4 + 2 + 2 = 9,$$

a contradiction. The P_4 table also gives $\lambda_4(Y) - 1 \geq 4$ whenever $4 \leq |Y| \leq 9$; such a part, together with the other nonempty part, would give $\lambda_5(X) \geq 1 + 4 + 2 = 7$. Hence each occupied piece has size 1, 2, or 3.

If both occupied pieces have size at least 2, then (2) gives $\lambda_5(X) \geq 1 + 3 + 3 = 7$. If their sizes are $(1, 2)$ or $(1, 3)$, then the contributions from the two parts are at least 2 and 3, so equality forces $m(X) = b(X) = 0$. Write the singleton part as $X_i = \{x\}$, and consider its mate $\mu(x)$. It lies outside V_i; it is not in X because $m(X) = 0$; and because $b(X) = 0$ it lies in $A = A_i \cup A_j$. Since $A_i \subseteq V_i$, this forces $\mu(x) \in A_j$, so $\mu(x)$ is internally adjacent to some $y \in X_j$. The same argument applied to that y (its mate lies outside V_j, is not in X, and lies in A) forces $\mu(y) \in A_i$, so $\mu(y)$ is internally adjacent to the unique vertex x of X_i. The four vertices

$$x,\ \mu(x),\ y,\ \mu(y)$$

are distinct: x and $\mu(y)$ lie in V_i and differ because $\mu(y) \in A_i$ and A_i is disjoint from X_i; likewise $\mu(x)$ and y lie in V_j and differ because A_j is disjoint from X_j. The edges $x\mu(x)$ and $y\mu(y)$ belong to the matching M, while $\mu(x)y$ and $\mu(y)x$ are internal edges, so these four vertices form a 4-cycle, contradicting Lemma 2.4.

The only remaining case is that both occupied pieces are singletons, say $X = \{x, y\}$. The recursive identity gives

$$6 = \lambda_5(X) = 1 + 2 + 2 + m(X) + b(X),$$

so $m(X) + b(X) = 1$. If $m(X) = 1$, then $y = \mu(x)$, so x and y are adjacent. If $b(X) = 1$, then exactly one of the two mates, say $\mu(x)$, lies in $X \cup A$; it is not in X because $m(X) = 0$, and as above it cannot lie in A_i, so $\mu(x) \in A_j$ is an internal neighbor of y. Being adjacent to x by its matching edge as well, it is a common neighbor of x and y. This proves the lemma. □

4.3. Proof of Theorem 1.3

Proof of Theorem 1.3(a). First take $n = 5$. Decompose P_5 into five copies of P_4, and let p be the number of nonempty sets X_i.

If $p = 1$, say $X = X_j$, then $|X_j| \geq 2$. The mate of every vertex of X lies in an empty part, so $m(X) = 0$ and, since the mates are distinct and avoid $X \cup A$, $b(X) = |X|$. Lemma 4.1 and (3) give

$$\lambda_5(X) = \lambda_4(X_j) + |X_j| \geq 6.$$

If $p \geq 3$, then (1) gives

$$\lambda_5(X) \geq 1 + 2p \geq 7 > 6.$$

It remains to consider $p = 2$. Let the two occupied parts be V_i and V_j. If $m(X) + b(X) \geq 1$, then (1) gives

$$\lambda_5(X) \geq 1 + 1 + 2 + 2 = 6.$$

Assume $m(X) + b(X) = 0$. Then no matching edge has both endpoints in X, and the mate of every vertex of X lies in the internal boundary $A_i \cup A_j$.

The mate of a vertex in V_i lies in the part indexed by its first symbol. Since a vertex in V_i cannot have first symbol i, every vertex of X_i must have first symbol j; similarly every vertex of X_j must have first symbol i. Hence

$$|X_i|, |X_j| \leq 3! = 6.$$

If one of X_i, X_j has size at least two, then (2) gives one contribution at least 3, while (1) gives the other contribution at least 2. Thus

$$\lambda_5(X) \geq 1 + 3 + 2 = 6.$$

If both X_i and X_j are singletons, say $X = \{x, y\}$, then $m(X) = 0$ and $b(X) = 0$ force $\mu(x)$ to be an internal neighbor of y and $\mu(y)$ to be an internal neighbor of x, exactly as in the proof of Lemma 4.2. The four distinct vertices

$$x,\ \mu(x),\ y,\ \mu(y)$$

then form a 4-cycle, contradicting Lemma 2.4. This last case is impossible. Therefore $\lambda_5(X) \geq 6$ for every $|X| \geq 2$.

Now let $n \geq 6$ and assume the theorem for P_{n-1}. Let

$$a = |\{i : |X_i| = 1\}|, \qquad q = |\{i : |X_i| \geq 2\}|.$$

For a singleton in P_{n-1},

$$\lambda_{n-1}(\{v\}) = (n-2), \qquad \lambda_{n-1}(\{v\}) - 1 = n - 3. \tag{4}$$

For every part of size at least two, the induction hypothesis gives

$$\lambda_{n-1}(X_i) - 1 \geq (2n-6) - 1 = 2n - 7. \tag{5}$$

If $q \geq 2$, then Lemma 4.1 and (5) give

$$\lambda_n(X) \geq 1 + 2(2n-7) = 4n - 13 \geq 2n - 4.$$

If $q = 1$ and $a \geq 1$, then (4) and (5) give

$$\lambda_n(X) \geq 1 + (2n-7) + (n-3) = 3n - 9 \geq 2n - 4.$$

If $q = 1$ and $a = 0$, then X lies wholly in one copy $P_n[V_j]$ and $|X| \geq 2$. The mate of every vertex of X lies in an empty copy, so $m(X) = 0$ and $b(X) = |X|$. Hence

$$\lambda_n(X) = \lambda_{n-1}(X_j) + |X_j| \geq (2n-6) + 2 = 2n - 4.$$

Finally suppose $q = 0$. Then X consists only of singletons in distinct parts, so $a = |X| \geq 2$. If $a \geq 3$, then

$$\lambda_n(X) \geq 1 + 3(n-3) = 3n - 8 \geq 2n - 4.$$

If $a = 2$, write $X = \{x, y\}$. If $y = \mu(x)$, then $m(X) = 1$. Otherwise $m(X) = 0$; if in addition $b(X) = 0$, then both mates lie in A, and as in the proof of Lemma 4.2 this forces $\mu(x)$ to be an internal neighbor of y and $\mu(y)$ an internal neighbor of x, giving the 4-cycle

$$x,\ \mu(x),\ y,\ \mu(y),\ x,$$

again contradicting Lemma 2.4. Therefore $m(X) + b(X) \geq 1$, and

$$\lambda_n(X) \geq 1 + 2(n-3) + 1 = 2n - 4.$$

This completes the induction. $\square$

Proof of Theorem 1.3(b). First suppose $X = \{u, v\}$. Put $d = n - 1$. If u and v are adjacent, then Lemma 2.3 gives no common neighbor, so

$$\lambda_n(X) = 1 + 2(d-1) - 2 + 1 = 2d - 2 = 2n - 4.$$

If u and v have a common neighbor, then Lemma 2.4 shows that the common neighbor is unique and u, v are nonadjacent by Lemma 2.3. Hence

$$\lambda_n(X) = 0 + (2d-1) - 2 + 1 = 2d - 2 = 2n - 4.$$

The converse is proved by induction on n. The case $n = 5$ is Lemma 4.2. Let $n \geq 6$, and assume the result for P_{n-1}. Use the notation of Lemma 4.1, and put

$$a = |\{i : |X_i| = 1\}|, \qquad q = |\{i : |X_i| \geq 2\}|.$$

If $q \geq 2$, then the proof of part (a) gives

$$\lambda_n(X) \geq 1 + 2(2n - 7) > 2n - 4.$$

If $q = 1$ and $a \geq 1$, then

$$\lambda_n(X) \geq 1 + (2n-7) + (n-3) > 2n - 4.$$

Thus equality can occur with $q = 1$ only when $a = 0$. In that case X lies inside a single copy of P_{n-1}, and

$$\lambda_n(X) = \lambda_{n-1}(X) + |X|.$$

Part (a) applied in the copy gives $\lambda_{n-1}(X) \geq 2n-6$, so equality in P_n forces $|X| = 2$ and $\lambda_{n-1}(X) = 2n-6$. By induction, the two vertices are adjacent or have a common neighbor inside the copy, and therefore also in P_n.

It remains to consider $q = 0$, so X consists of $a = |X|$ singletons in distinct copies. If $a \geq 3$, the proof of part (a) gives

$$\lambda_n(X) \geq 1 + 3(n-3) > 2n-4.$$

Hence equality forces $a = 2$, say $X = \{x, y\}$. If $y = \mu(x)$, then x and y are adjacent. Otherwise $m(X) = 0$, and the recursive identity gives

$$2n-4 = \lambda_n(X) = 2n-5+b(X),$$

so $b(X) = 1$. Thus exactly one of the two mates lies in $X \cup A$; as in the proof of Lemma 4.2, it lies in the internal boundary of the other singleton's part and is therefore a common neighbor of x and y. This completes the induction. $\square$

5. The exact values

Proposition 5.1 (Local induced-path upper bound). *Let G be a d-regular graph with $d \geq 3$. If G contains an induced path x–s–y, then*

$$\mathrm{cfsmp}(G) \leq 2d-2.$$

Proof. Delete every edge incident to x except xs, and every edge incident to y except ys. Since the path is induced, the two deleted edge sets are disjoint, so this uses $2(d-1) = 2d-2$ faults.

In the remaining graph H, the vertices x and y are leaves adjacent to s. Thus $H - \{s\}$ has at least two isolated vertices, and Lemma 2.1 shows that H has no fractional perfect matching.

No isolated vertex is created. The vertices x and y remain adjacent to s, the vertex s remains adjacent to both of them, and every other vertex loses at most two incident edges from a graph of degree at least three. Hence the deleted edge set is a conditional fractional strong matching preclusion set. $\square$

Proposition 5.2. *For every $n \geq 4$,*

$$\mathrm{cfsmp}(P_n) \leq 2n-4.$$

Proof. Let

$$s = 123\cdots n, \qquad x = 2134\cdots n, \qquad y = 3214\cdots n.$$

Then x and y are both adjacent to s. They are not adjacent to each other: reversing a prefix of $2134\cdots n$ gives $1234\cdots n$ for $k=2$, gives $3124\cdots n$ for $k=3$, and for $k\geq 4$ does not give $3214\cdots n$. Hence x–s–y is an induced path in P_n. Since P_n is $(n-1)$-regular, Proposition 5.1 gives the result. $\square$

Proof of Theorem 1.1. For $n \geq 5$, the upper bound is Proposition 5.2. For the lower bound, let F be any conditional fractional strong matching preclusion set. By Lemma 3.1, there is $X \subseteq V(P_n)$ with $|X| \geq 2$ and $|F| \geq \lambda_n(X)$. Theorem 1.3(a) gives $|F| \geq 2n-4$. Hence $\mathrm{cfsmp}(P_n) = 2n-4$.

It remains to discuss P_4. The table in Section 4 shows $\lambda_4(X) \geq 3$ for every nonempty X, so Lemma 3.1 gives $\mathrm{cfsmp}(P_4) \geq 3$.

For the matching upper bound, write permutations in one-line notation. Let F consist of the vertex 1234 and the two edges

$$1324\text{–}4231, \qquad 2143\text{–}3412.$$

Let

$$S = \{1243, 1432, 2314, 2413, 2431, 3124, 3142, 3241,\\ 4123, 4312, 4321\}$$

and

$$X = \{1324, 1342, 1423, 2134, 2143, 2341, 3214, 3412,\\ 3421, 4132, 4213, 4231\}.$$

Appendix A lists the three neighbors of every vertex of X and shows that each one is either deleted by F or lies in S; hence every vertex in X is isolated in $(P_4 - F) - S$. Since $|X| = 12 > |S| = 11$, Lemma 2.1 implies that $P_4 - F$ has no fractional perfect matching. The same data shows that $P_4 - F$ has no isolated vertex. Thus $\mathrm{cfsmp}(P_4) \leq 3$, and the lower bound above gives $\mathrm{cfsmp}(P_4) = 3$. $\square$

6. Classification of the optimal fault sets

Throughout this section, $n \geq 5$, so P_n is $(n-1)$-regular with $n-1 \geq 4$. We first check that the sets $F(u, w, v; Z)$ of Theorem 1.2 never isolate a vertex; the argument uses only the girth structure and works in any graph of the same local shape.

Lemma 6.1 (Mixed fault non-isolation)**.** *Let G be a triangle-free and C_4-free graph with minimum degree at least* 4*. Let u–w–v be an induced path of length two in G, and put*

$$B = (N(u) \cup N(v)) \setminus \{w\}.$$

For any subset $Z \subseteq B$, let

$$F = Z \ \cup \ \{xy : \ y \in B \setminus Z, \ x \in \{u, v\}, \ xy \in E(G)\}.$$

Then $G - F$ has no isolated vertex.

Proof. Put $B_u = N(u) \setminus \{w\}$ and $B_v = N(v) \setminus \{w\}$. The vertices u and v remain adjacent to w, and w remains adjacent to both of them.

Let $y \in B_u \setminus Z$. The vertex y loses the edge to u, and it may lose neighbors that lie in Z. It has no neighbor in B_u, since such a neighbor would form a triangle with u. It has at most one neighbor in B_v, since two such neighbors would give a 4-cycle through y and v. Thus y loses at most two incident adjacencies. The same argument applies to vertices in $B_v \setminus Z$.

Finally, let $y \notin B \cup \{u, v, w\}$ be a surviving vertex. No edge fault is incident with y. By C_4-freeness, the vertex y has at most one neighbor in B_u and at most one neighbor in B_v, so it loses at most two neighbors to vertex faults in Z. Since every vertex has degree at least 4, no surviving vertex is isolated in $G - F$. $\square$

Proof of Theorem 1.2. Let $d = n - 1$. Since P_n is triangle-free and C_4-free, the vertices u and v are nonadjacent and have the unique common neighbor w. Hence each vertex of B is adjacent to exactly one of u, v, and

$$|B| = 2(d - 1) = 2n - 4.$$

Therefore $|F(u, w, v; Z)| = 2n - 4$. In $P_n - F(u, w, v; Z)$, the vertices u and v are leaves adjacent to w. After deleting w, both become isolated, so Lemma 2.1 shows that the faulty graph has no fractional perfect matching. Since P_n is triangle-free, C_4-free, and has minimum degree $d = n - 1 \geq 4$, Lemma 6.1 shows that no isolated vertex is created. Hence every $F(u, w, v; Z)$ is a CFSMP set, and it is optimal by Theorem 1.1.

Conversely, let F be an optimal CFSMP set, and put $H = P_n - F$. By Lemma 2.1, there are $S \subseteq V(H)$ and a set X of isolated vertices in $H - S$ with $|X| > |S|$. Lemma 3.1, Theorem 1.3(a), and Theorem 1.1 give

$$2n - 4 = |F| \geq \lambda_n(X) \geq 2n - 4.$$

Thus $\lambda_n(X) = 2n - 4$, and Theorem 1.3(b) gives $X = \{u, v\}$ where u, v are adjacent or have a common neighbor. Since $|F| = \lambda_n(X)$, Lemma 3.2 gives $|S| = 1$ and $S \subseteq \partial X$.

Write $S = \{w\}$. The vertices u and v are isolated in $H - \{w\}$ but not isolated in H, so both are adjacent to w in H. If u and v were adjacent, then u, v, w would form a triangle in P_n, contradicting Lemma 2.3. Hence u and v are nonadjacent and have the common neighbor w.

By Lemma 3.2, every boundary vertex of $X = \{u, v\}$ except w is accounted for by exactly one fault: either the boundary vertex itself is deleted, or the unique edge from that boundary vertex to u or v is deleted. Lemmas 2.3 and 2.4 make this edge unique for every boundary vertex outside w. The equality lemma also shows that there are no additional faults. Thus $F = F(u, w, v; Z)$ for the set Z of boundary vertices deleted by F.

In particular, every optimal set leaves two leaves u, v adjacent to the same vertex w, so every optimal set is trivial in the sense of Gupta, Cheng, and Lipták. □

Remark 6.2 (A mixed optimal set in P_5)**.** *Let*

$$w = 12345, \qquad u = 21345, \qquad v = 32145.$$

Then u–w–v is an induced path. The boundary set $B = (N(u) \cup N(v)) \setminus \{w\}$ is

$$\{31245, 43125, 54312, 23145, 41235, 54123\}.$$

Taking $Z = \{31245, 23145\}$ gives the optimal mixed fault set

$$\begin{aligned} F(u, w, v; Z) = \{&31245, 23145,\\ &21345\text{–}43125,\ 21345\text{–}54312,\\ &32145\text{–}41235,\ 32145\text{–}54123\}. \end{aligned}$$

Thus optimal sets are not only pure edge-deletion sets; boundary vertex faults may replace the corresponding boundary edge faults.

7. Pancake-like compositions

Gupta, Cheng, and Lipták proved the following composition theorem, which propagates both supermatchedness properties through one level of recursive assembly.

Theorem 7.1 (Gupta–Cheng–Lipták [7, Theorem 7])**.** *Let $k \geq 4$ and $N \geq 3$, and let $H_1, \ldots, H_N$ be $(k-1)$-regular graphs, each of girth at least* 5*. Let G be a k-regular graph obtained from the disjoint union of $H_1, \ldots, H_N$ by adding edges between the subgraphs so that (i) every vertex is incident with exactly one added edge, and (ii) the added edges at the endpoints of any edge of any H_i lead to different subgraphs. If every H_i is fractional strongly super matched and conditionally fractional strongly super matched, then G is fractional strongly super matched and conditionally fractional strongly super matched.*

Theorem 7.1 is Theorem 7 of the conference version [7], where its proof is omitted for reasons of space; the journal version [8] is the full account. We have rephrased the two cross-edge conditions in our notation: condition (ii) is their requirement that the cross neighbours of two adjacent vertices in any H_i lie in different subgraphs.

For ordinary pancake building blocks, the fractional strong hypothesis is supplied by Ma, Mao, Cheng, and Melekian [15], and the conditional hypothesis is Theorem 1.2. Recall that the burnt pancake graph B_n is the Cayley graph of the group of signed permutations of n symbols generated by the n signed prefix reversals; it is n-regular and has girth 8 [6].

Corollary 7.2. *Let $m \geq 5$ and $N \geq 3$, and let G be an m-regular graph obtained from N disjoint graphs $H_1, \ldots, H_N$, each isomorphic to the pancake graph P_m or to the burnt pancake graph B_{m-1}, by adding edges subject to conditions* (i) *and* (ii) *of Theorem 7.1. Then G is fractional strongly super matched and conditionally fractional strongly super matched.*

Proof. Each H_i is $(m-1)$-regular with $m \geq 5$. The pancake graph P_m is triangle-free and C_4-free by Lemmas 2.3 and 2.4, hence of girth at least 5; it is fractional strongly super matched for $m \geq 5$ [15] and conditionally fractional strongly super matched by Theorem 1.2. The burnt pancake graph B_{m-1} has girth 8 [6]; it is fractional strongly super matched for $m-1 \geq 3$ [15] and conditionally fractional strongly super matched for $m-1 \geq 3$ [8]. Theorem 7.1 with $k = m$ gives the conclusion. □

Since P_n decomposes for $n \geq 6$ into n copies of P_{n-1} satisfying conditions (i) and (ii), Corollary 7.2 applies inductively: the case $n = 5$ of Theorem 1.2 alone already propagates supermatchedness to every P_n with $n \geq 6$. The direct induction of Sections 4–6 yields more, namely the exact list of optimal fault sets, which does not follow from supermatchedness alone.

8. Concluding remarks

The value $2n-4$ should be contrasted with the regular bipartite case, where the invariant collapses for an elementary reason.

Proposition 8.1. *Let G be a d-regular bipartite graph with $d \geq 2$. Then* $\operatorname{cfsmp}(G) = 1$.

Proof. Let A and B be the two parts. Since $d|A| = |E(G)| = d|B|$, the two parts have equal size.

Assigning weight $1/d$ to every edge gives a fractional perfect matching, so the empty fault set does not work. Deleting one vertex leaves no isolated vertex because $d \geq 2$. But the remaining bipartite graph has part sizes differing by one, and therefore cannot have a fractional perfect matching: summing the fractional matching constraints over the two parts would give two different totals for the same total edge weight. ☐

Proposition 8.1 also dispatches the small pancake graphs excluded from Theorem 1.1. The graph P_3 is a 6-cycle, hence 2-regular bipartite, so $\operatorname{cfsmp}(P_3) = 1$. The graphs P_1 and P_2 admit no conditional fractional strong matching preclusion set at all, so cfsmp is undefined for them: in $P_2 \cong K_2$, a fault set leaving no isolated vertex leaves either P_2 itself or the graph with no vertices, and both have fractional perfect matchings (the latter vacuously); in P_1, the empty fault set leaves an isolated vertex and the full one leaves the empty graph.

Bipartite Cayley graphs of symmetric groups generated by prefix reversals do occur; the following family is introduced here purely as a contrast object. Let $n \equiv 3 \pmod 4$, and define the odd-prefix pancake graph

$$\mathrm{OP}_n = \operatorname{Cay}\bigl(\mathfrak{S}_n, \{r_k : 2 \leq k \leq n,\ k \equiv 2, 3 \pmod 4\}\bigr).$$

Corollary 8.2. *For every $n \geq 3$ with $n \equiv 3$* (mod 4)*,*

$$\operatorname{cfsmp}(\mathrm{OP}_n) = 1.$$

Proof. The prefix reversal r_k has sign $(-1)^{k(k-1)/2}$, so the generators in the definition of OP_n are odd permutations. Hence OP_n is bipartite with the even and odd permutations as its two parts, and these parts have equal size. The degree is $(n+1)/2 \geq 2$. Proposition 8.1 applies. ☐

Remark 8.3. *The local upper bound in Proposition 5.1 need not be sharp in small dimensions. The graph P_4 is* 3*-regular, so the local construction* *gives only* cfsmp$(P_4) \leq 4$*. Theorem 1.1 gives the exact value* cfsmp$(P_4) = 3$*, showing that a naive universal equality* cfsmp$(G) = 2d - 2$ *cannot hold even within the pancake family.*

Appendix A. A certificate for the exceptional fault set in P_4

This appendix makes the exceptional P_4 construction in the proof of Theorem 1.1 checkable by hand. Recall the fault set F consisting of the vertex 1234 and the edges 1324–4231 and 2143–3412, and the sets S (11 vertices) and X (12 vertices) listed there; together with the deleted vertex 1234, they partition $V(P_4)$.

Each vertex of P_4 has exactly three neighbors, obtained by the prefix reversals r_2, r_3, and r_4. The table lists the three neighbors of every vertex of X. A superscript v marks the deleted vertex 1234; a superscript e marks a neighbor whose connecting edge is one of the two edge faults; every unmarked entry belongs to S.

$x \in X$	r_2x	r_3x	r_4x
1324	3124	2314	4231^e
1342	3142	4312	2431
1423	4123	2413	3241
2134	1234^v	3124	4312
2143	1243	4123	3412^e
2341	3241	4321	1432
3214	2314	1234^v	4123
3412	4312	1432	2143^e
3421	4321	2431	1243
4132	1432	3142	2314
4213	2413	1243	3124
4231	2431	3241	1324^e

Three facts can be read off directly. First, every neighbor of a vertex of X that survives in $P_4 - F$ lies in S, so every vertex of X is isolated in $(P_4 - F) - S$. Second, every vertex of X retains at least one neighbor in S, so no vertex of X is isolated in $P_4 - F$. Third, each of the 11 vertices of S appears in the table as an unmarked entry, so each has a surviving neighbor in X and is not isolated in $P_4 - F$. Since $V(P_4) = \{1234\} \cup S \cup X$, the graph $P_4 - F$ has no isolated vertex.

References

[1] M. Abdallah and E. Cheng, Conditional strong matching preclusion of the pancake graph, *International Journal of Parallel, Emergent and Distributed Systems* 38 (2023), 1–13. DOI: 10.1080/17445760.2022.2066102.

[2] R. C. Brigham, F. Harary, E. C. Violin, and J. Yellen, Perfect-matching preclusion, *Congressus Numerantium* 174 (2005), 185–192.

[3] E. Cheng, P. Hu, R. Jia, L. Lipták, B. Scholten, and J. R. Voss, Matching preclusion and conditional matching preclusion for pancake and burnt pancake graphs, *International Journal of Parallel, Emergent and Distributed Systems* 29 (2014), 499–512. DOI: 10.1080/17445760.2013.839444.

[4] E. Cheng, L. Lipták, and L. Mazza, Higher order matching preclusion for regular interconnection networks, *Discrete Applied Mathematics* 373 (2025), 107–124. DOI: 10.1016/j.dam.2025.04.042.

[5] E. Cheng, L. Lesniak, M. J. Lipman, and L. Lipták, Conditional matching preclusion sets, *Information Sciences* 179 (2009), 1092–1101. DOI: 10.1016/j.ins.2008.10.029.

[6] P. E. C. Compeau, Girth of pancake graphs, *Discrete Applied Mathematics* 159 (2011), 1641–1645.

[7] S. Gupta, E. Cheng, and L. Lipták, Conditional fractional matching preclusion for burnt pancake graphs and pancake-like graphs (extended abstract), in: C.-Y. Chen, W.-K. Hon, L.-J. Hung, and C.-W. Lee (eds.), *Computing and Combinatorics (COCOON 2021)*, Lecture Notes in Computer Science 13025, Springer, Cham, 2021, pp. 425–435.

[8] S. Gupta, E. Cheng, and L. Lipták, Conditional fractional matching preclusion for burnt pancake graphs and pancake-like graphs, *International Journal of Computer Mathematics: Computer Systems Theory* 7 (2022), 207–222. DOI: 10.1080/23799927.2022.2110159.

[9] X. Hu, X. Ren, and W. Yang, Conditional k-matching preclusion for n-dimensional torus networks, *Discrete Applied Mathematics* 353 (2024), 181–190. DOI: 10.1016/j.dam.2024.04.026.

[10] X. Hu, Y. Tian, J. Meng, and W. Yang, Conditional fractional matching preclusion of n-dimensional torus networks, *Discrete Applied Mathematics* 293 (2021), 157–165. DOI: 10.1016/j.dam.2021.01.011.

[11] W. Li, Y. Liu, Y. Li, E. Cheng, and Y. Mao, Conditional fractional matching preclusion number of graphs, *International Journal of Foundations of Computer Science* 36 (2025), 143–159. DOI: 10.1142/S0129054124500114.

[12] R. Lin and H. Zhang, Fractional matching preclusion number of graphs and the perfect matching polytope, *Journal of Combinatorial Optimization* 39 (2020), 915–932. DOI: 10.1007/s10878-020-00530-2.

[13] Y. Liu and W. Liu, Fractional matching preclusion of graphs, *Journal of Combinatorial Optimization* 34 (2017), 522–533. DOI: 10.1007/s10878-016-0077-x.

[14] Y. Luan, M. Lu, and Y. Zhang, Fractional matching preclusion numbers of Cartesian product graphs, *Discrete Applied Mathematics* 338 (2023), 100–112. DOI: 10.1016/j.dam.2023.05.021.

[15] T. Ma, Y. Mao, E. Cheng, and C. Melekian, Fractional matching preclusion for (burnt) pancake graphs, in *2018 15th International Symposium on Pervasive Systems, Algorithms and Networks (I-SPAN)*, IEEE, 2018, pp. 133–141. DOI: 10.1109/I-SPAN.2018.00030.

[16] Y. Mao and E. Cheng, A concise survey of matching preclusion in interconnection networks, *Journal of Interconnection Networks* 19 (2019), 1940006. DOI: 10.1142/S0219265919400061.

[17] J.-H. Park and I. Ihm, Strong matching preclusion, *Theoretical Computer Science* 412 (2011), 6409–6419. DOI: 10.1016/j.tcs.2011.07.028.

[18] E. R. Scheinerman and D. H. Ullman, *Fractional Graph Theory: A Rational Approach to the Theory of Graphs*, Wiley, 1997.

[19] H. Wu, L. Ba, and H. Zhang, Integer k-matching preclusion of some interconnection networks, *Theoretical Computer Science* 992 (2024), Article 114449. DOI: 10.1016/j.tcs.2024.114449.

[20] J. Zou, Y. Mao, Z. Wang, and E. Cheng, Fractional matching preclusion number of graphs, *Discrete Applied Mathematics* 311 (2022), 142–153. DOI: 10.1016/j.dam.2022.01.014.